CURVATURE ESTIMATE AND THE STABILITY OF MINIMAL HYPERSURFACES

SHENGRONG WU

ABSTRACT. This article introduces the theory of minimal surface and some important properties and applications about curvature estimate. In the second part we introduce the proof of Bernstein's theorem. In the third part we focus on the minimal surface equation. In the forth part, we used the curvature estimate theorems to prove several stability theorems.

CONTENTS

1. INTRODUCTION

In Riemannian geometry, we have defined the second fundamental form and the mean curvature. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇, and Σ be a submanifold of M. The second fundamental form of Σ is a vector-valued bilinear form $A(X,Y) = (\nabla_X Y)^N$, where V^N denotes the normal component of a tangent vector V. The mean curvature vector H at x is

$$H = \sum_{i=1}^{k} A(E_i, E_j),$$

where $E_i, i = 1, ..., k$ is an orthonormal basis for $T_x\Sigma$.

Let $F : \Sigma \times (-\epsilon, \epsilon) \to M$ be a variation of Σ with compact surpport. Let $\Sigma_t = F_t(\Sigma)$, and x^i the local coordinates on Σ.

$$\text{Vol}(\Sigma_t) = \int_\Sigma \sqrt{det(g(dF(\frac{\partial}{\partial x^i}), dF(\frac{\partial}{\partial x^j})))}dx^1 \wedge \cdots \wedge dx^k.$$

Set

$$g_{ij}(t) = g(dF(\frac{\partial}{\partial x^i}), dF(\frac{\partial}{\partial x^j})),$$

and

$$v(t) = \sqrt{det(g_{ij}(t))det(g^{ij}(0))}.$$

Then

$$\frac{d}{dt}_{t=0} Vol(\Sigma_t) = \int_\Sigma \frac{d}{dt}_{t=0} v(t)dV.$$

$$\begin{aligned}\frac{d}{dt}_{t=0}v(t) &= \frac{1}{2}(det(g_{ij}(t)))^{-\frac{1}{2}}\sqrt{det(g^{ij}(0))} \quad \frac{d}{dt}_{t=0}det(g_{ij}(t))\\ &= \frac{1}{2}(det(g_{ij}(t)))^{-\frac{1}{2}}\sqrt{det(g^{ij}(0))} \quad det(g_{ij}(t))Tr[(g^{ij}(t))(\frac{d}{dt}g_{ij}(t))]\\ &= \frac{1}{2}Tr[(g^{ij}(0))(g(\nabla_{F_t}F_{x^i},F_{x^j})+g(\nabla_{F_t}F_{x^j},F_{x^i}))].\end{aligned}$$

Let x^i be normal coordinates at x, then

$$\frac{d}{dt}_{t=0}v(t) = \sum_{i=1}^{k} g(\nabla_{F_t}F_{x^i},F_{x^i}) = \sum_{i=0}^{k} g(\nabla_{F_{x^i}}F_t,F_{x^i}) = div_\Sigma F_t$$

Choose a orthonormal basis $N_l, l = 1,\cdots,n-k$ for NxM, so

$$F_t = g(F_t,N_l)N_l + F_t^T,$$

and

$$\begin{aligned}\frac{d}{dt}_{t=0}v(t) &= \langle\nabla_{F_{x^i}}N_l,F_{x^i}\rangle\langle F_t,N_l\rangle + div_\Sigma F_t^T\\ &= -\langle N_l,\nabla_{F_{x^i}}F_{x^i}\rangle\langle F_t,N_l\rangle + div_\Sigma F_t^T\\ &= -\langle F_t,A(F_{x^i},F_{x^i})\rangle + div_\Sigma F_t^T\\ &= -\langle F_t,H\rangle + div_\Sigma F_t.\end{aligned}$$

Therefore,

$$\frac{d}{dt}_{t=0}\mathrm{Vol}(\Sigma_t) = -\int_\Sigma \langle F_t,H\rangle dV. \tag{1.1}$$

Definition 1.1. A submanifold $\Sigma \subset (M,g)$ is called *minimal* if $H = 0$.

There is an important lemma, which connects minimal surfaces and harmonic functions.

Lemma 1.1. *$\Sigma^k \subset \mathbb{R}^n$ is minimal if and only if the restrictions of the coordinate functions of $\mathbb{R}^n$ to Σ are harmonic.*

Now assume Σ is minimal, we next compute the second derivative of $Vol(\Sigma_t)$. For convenience, we assume that $F_t^T = 0$.

$$\frac{d^2}{dt^2}_{t=0}Vol(\Sigma_t) = \int_\Sigma \frac{d^2}{dt^2}v(t)dV,$$

and in the preceeding computation we obtained

$$2\frac{d}{dt}v(t) = Tr[(g'_{ij}(t))(g^{ij}(t))]v(t).$$

Choose normal coordinates at x, then

$$\begin{aligned}2\frac{d^2}{dt^2}_{t=0}v(t) &= Tr[g''_{ij}(0)] + Tr[(g'_{ij}(0))(g^{ij\prime}(0))] + \frac{1}{2}Tr[(g'_{ij}(0))]^2\\ &= Tr[g''_{ij}(0)] - Tr[(g'_{ij}(0))(g'_{ij}(0))] + \frac{1}{2}Tr[(g'_{ij}(0))]^2\end{aligned}$$

$$\begin{aligned}
Tr[g'_{ij}] &= 2\langle \nabla_{F_{x^i}} F_t, F_{x^i}\rangle \\
&= 2\langle \nabla_{F_{x^i}} F_t^T, F_{x^i}\rangle + 2\langle \nabla_{F_{x^i}} F_t^N, F_{x^i}\rangle \\
&= -2\langle \nabla_{F_{x^i}} F_{x_i}, F_t^N\rangle + 2\langle \nabla_{F_{x^i}} F_t^T, F_{x^i}\rangle = 0
\end{aligned}$$

$$\begin{aligned}
Tr[(g''_{ij}(0))] &= 2\sum_{i=1}^{k}\langle F_{x^i tt}, F_{x^i}\rangle + 2\sum_{i=1}^{k}\langle F_{x^i t}, F_{x^i t}\rangle \\
&= 2\langle F_{tx^i t}, F_{x^i}\rangle + 2\langle F_{x^i t}, F_{x^i t}\rangle \\
&= 2\langle R_M(F_{x^i}, F_t)F_t, F_{x^i}\rangle + 2\langle F_{ttx^i}, F_{x^i}\rangle + 2\langle F_{x^i t}, F_{x^i t}\rangle \\
&= 2R(F_{x^i}, F_t, F_{x^i}, F_t) + 2div_\Sigma F_{tt} + 2\langle F_{x^i t}, F_{x^i t}\rangle .
\end{aligned}$$

Note that

$$\begin{aligned}
\langle F_{x^i t}, F_{x^i t}\rangle &= \langle F_{x^i t}, F_{x^j}\rangle\langle F_{x^i t}, F_{x^j}\rangle + \langle F_{x^i t}, F_{x^i t}^N\rangle \\
&= |\langle A(\cdot,\cdot), F_t\rangle|^2 + |grad_\Sigma^N F_t|^2,
\end{aligned}$$

and we get

$$Tr[(g'_{ij}(0))] = 2R(F_{x^i}, F_t, F_{x^i}, F_t) + 2div_\Sigma F_{tt} + |2\langle A(\cdot,\cdot), F_t\rangle|^2 + 2|\nabla_\Sigma^N F_t|^2 .$$

Since

$$\langle F_{tx^i}, F_{x^j}\rangle + \langle F_{x^i}, F_{tx^j}\rangle = -2\langle F_t, F_{x^i x^j}\rangle = -2\langle A(F_{x^i}, F_{x^j}), F_t\rangle ,$$

$$\begin{aligned}
Tr[(g'_{ij}(0))(g'_{ij}(0))] &= (\langle F_{tx^i}, F_{x^j}\rangle + \langle F_{x^i}, F_{tx^j}\rangle)(\langle F_{tx^j}, F_{x^i}\rangle + \langle F_{x^j}, F_{tx^i}\rangle) \\
&= 4|\langle A(\cdot,\cdot), F_t\rangle|^2 .
\end{aligned}$$

Combining the equations, we have

$$\frac{d^2}{dt^2}_{t=0} v = -|\langle A(\cdot,\cdot), F_t\rangle|^2 + |\nabla_\Sigma^N F_t|^2 + \sum_{i=1}^{k} R(F_{x^i}, F_t, F_{x^i}, F_t) + div_\Sigma F_{tt} .$$

Define an self-sdjoint operator L by

$$LX = \Delta_\Sigma^N X + \sum_{i=1}^{k} R_M(E_i, X)E_i + \sum_{i,j=1}^{k}\langle A(E_i, E_j), X\rangle A(E_i, E_j),$$

where $E_i, i = 1, \cdots, k$ is the locally orthonormal frame on Σ , and the second derivative of the volume can be written in the form

$$\frac{d^2}{dt^2}_{t=0}\mathrm{Vol}(\Sigma_t) = -\int_\Sigma \langle F_t, LF_t\rangle . \tag{1.2}$$

Definition 1.2. The operator L is called *stability operator* and a minimal submanifold $\Sigma \subset M$ is *stable* if

$$-\int \langle X, LX\rangle \geq 0$$

holds for all normal vector field X with compact support. Moreover, a oriented, immersed minimal hypersurfaces $\Sigma^{n-1} \subset \mathbb{R}^n$ is stable if

$$\int_\Sigma |A|^2 f^2 \leq \int_\Sigma |\nabla_\Sigma f|^2,$$

for any $f \in C_0^\infty(\Sigma)$.

In many cases, we may assume Σ is a hypersurface with trivial normal bundle. So we can identify a normal vector field X with a function $f = \langle X, N\rangle$ and in this way,

$$Lf = \Delta_\Sigma f + |A|^2 f + Ric_M(N,N)f. \tag{1.3}$$

From the computation, we immediately obtain the following lemma.

Lemma 1.2. *(The Stability Inequality) $\Sigma^{n-1} \subset M^n$ is a stable minimal hypersurface with trivial normal bundle, then for all $f \in C_0^1(\Sigma)$*

$$\int_\Sigma (Ric_M(N,N) + |A|^2) f^2 \leq \int_\Sigma |\nabla_\Sigma f|^2, \tag{1.4}$$

where N is the unit normal vector field.

Because $-L$ can be seen as a symmetric functor on $C_0^\infty(D)$ for each domain $D \subset \Sigma$, we can study its first eigenvalues to find the stability of Σ. Define

$$\lambda_1(D,L) = \inf\{-\int_D fLf : f \in C_0^\infty(D), \int_D f^2 = 1\}$$

We can apply a general result for the operator $\Delta - c$ to the stablility equation to obtain the following lemma.

Lemma 1.3. *(D.Fisher-Colbrie and R.Schoen* (cf.[?CDR1])*) If Σ is a complete non-compact oriented minimal hypersurface, then the following statements are equivalent:*

- *$\lambda_1(D,L) \geq 0$ for every bounded domain $D \subset \Sigma$.*
- *$\lambda_1(D,L) > 0$ for every bounded domain $D \subset \Sigma$.*
- *There exists a positive function u with $Lu = 0$.*

Remark 1.1. There exists a positive function u compactly surpported on D if and only if D is a stable domain.

There are some famous conjectures about the structure of minimal hypersurfaces.

Conjecture 1.1. *(R.Schoen) If $\Sigma^3 \subset \mathbb{R}^4$ is a complete, oriented, immersed, stable minimal hypersurface, then Σ is flat.*

This conjecture has been proved in 2021 by Otis Chodosh and Chao Li, and in the third part we will sketch its proof.

Conjecture 1.2. *(Bernstein) Suppose $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is a smooth function such that its graph is minimal in $\mathbb{R}^n$, then f is linear.*

Bernstein has proved the case $n = 3$, and De Giorgi showed that if every area-minimizing cone in $\mathbb{R}^k$ is planar then Bernstein's conjecture holds for the case $n = k$. In this way, he proved the case $n = 4$, and Almgren proved the case $n = 5$. Simons extends Bernstein's theorem to $\mathbb{R}^8$. But in fact, the conjecture fails when $n \geq 9$.

2. The Minimal Surface Equation and Bernstein's Theorem

Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth function with non-zero gradient, and the question is when its graph is a minimal hypersurface in $\mathbb{R}^n$. Let $\Sigma_f = \{(x^1, x^2, \cdots, x^{n-1}, f(x) : x^1, \cdots, x^{n-1} \in \mathbb{R})\}$ be the graph of f. Consider the variation $f + tg$ where $g \in C_0^\infty(\mathbb{R}^{n-1})$,

$$\begin{aligned}\frac{d}{dt}_{t=0} Vol(\Sigma_{f+tg}) &= \int_{\mathbb{R}^{n-1}} \frac{d}{dt}_{t=0} \sqrt{1 + |\nabla f + t\nabla g|} dV \\ &= -\int_{\mathbb{R}^{n-1}} g \cdot \operatorname{div}(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}).\end{aligned}$$

The equation

$$\mathcal{M}f = \operatorname{div}(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}) = 0, \tag{2.1}$$

is called the *minimal surface equation*.

According to [?CM1], we introduce a useful curvature estimate to prove Bernstein theorem when $n = 3$.

Lemma 2.1. *If $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ satisfies the minimal surface equation,then for all $\eta \in C_0^1(U \times \mathbb{R})$,*

$$\int_{\Sigma_f} |A|^2 \eta^2 \geq C(f) \int_{\Sigma_f} |\nabla_{\Sigma_f} \eta|^2 \tag{2.2}$$

Sketch of Proof. Consider the Gauss map $g : \Sigma_f \to \S^2$. The image of g is contractible so the volume form ω on S^2 is exact on $g(\Sigma_f)$. Let $d\alpha = \omega$. We can find a $C = C(\alpha)$ such that $|g * \alpha| \leq C|A|$. Since $|A|^2 = -2K$, we have

$$\begin{aligned}\int_{\Sigma_f} \eta^2 |A|^2 d\sigma &= -4 \int \eta d\eta \wedge g * \alpha \\ &\leq C \int \eta |\nabla_{\Sigma_f} \eta| |A| d\sigma \leq C \sqrt{\int \eta^2 |A|^2} \sqrt{|\nabla_{\Sigma_f}|^2}.\end{aligned}$$

Theorem 2.1. *(Bernstein) The graph of an entire solution to the minimal surface equation in $\mathbb{R}^3$ is flat.*

Proof. Consider the logarithmic cutoff funcion,

$$\eta(x) = \begin{cases} 1, & if \quad r \leq \sqrt{R} \\ 2 - 2\log(r)/\log R, & if \quad r \in (\sqrt{R}, R] \\ 0, & if \quad r > R. \end{cases}$$

We have

$$|\nabla_{\Sigma_f} \eta| \leq \frac{C}{r \log R}.$$

Applying the preceding lemma, we get

$$\begin{aligned}\int_{B_{\sqrt{R}}\cap\Sigma_f}|A|^2 &\le \int_{\Sigma_f}|A|^2\eta^2\\ &\le C\int_{\Sigma_f}|\nabla_{\Sigma_f}\eta|^2\\ &\le \frac{C}{(\log R)^2}\int_{B_R\cap\Sigma_f}r^{-2}\\ &\le \frac{C}{(\log R)^2}\sum_{k=(\log R)/2}^{\log R}\int_{(B_{e^k}\setminus B_{e^{k-1}})\cap\Sigma_f}r^{-2}\\ &\le \frac{C}{(\log R)^2}\sum_{k=(\log R)/2}^{\log R}Ce^2\\ &\le \frac{C}{\log R}.\end{aligned}$$

Letting $R\to\infty$, we get $|A|=0$. Therefore, Σ_f is flat.

As for the high dimensional case, de Giorgi improved in [?GE1] that if there is no minimal cones in $\mathbb{R}^{n-1}$, then the Bernstein theorem can be extended to $\mathbb{R}^n$. In this way, Bernstein conjecture was solved when $n=4,5,8$. In [?BGG1], Bombieri, de Giorgi and Giusti gave the counterexample for $n\ge 9$.

Definition 2.1. $\Sigma^{k-1}\subset S^{n-1}$ is a submanifold, then the *cone* over Σ is defined by

$$C(\Sigma)=\{\lambda x:\lambda\in\mathbb{R}\setminus\{0\},x\in\Sigma\}$$

.

Proposition 2.1. *The following statements are equivalent:*

(1) *$C(\Sigma)$ is minimal in $\mathbb{R}^n$.*
(2) *Σ is minimal in S^{n-1}.*
(3) *$\Delta_\Sigma x^i+(k-1)x^i=0, i=1,\cdots,n$, where $(x^1,\cdots,x^n)$ are the coordinate functions of $\mathbb{R}^n$ restricted on Σ.*

Proof. Note that the mean curvature of $\Sigma\subset\mathbb{R}^n$ is equal to

$$-\Delta_\Sigma x=-(\Delta_\Sigma x^1,\cdots,\Delta_\Sigma x^n),$$

so $\Sigma\subset S^{n-1}$ is minimal if and only if $\Delta_\Sigma x$ is normal to S^{n-1}. Therefore there is a function f such that $\Delta_\Sigma x=f\cdot x$. Exactly, $f=-(k-1)$ since

$$0=\Delta_\Sigma|x|^2=2f+2(k-1).$$

For $u\in C^\infty(C(\Sigma))$, it can be written as $u(x)=u(r,\theta)$.

$$\nabla_{C(\Sigma)}u=\frac{1}{r}\nabla_\Sigma u(r^{-1}\cdot)+\frac{\partial u}{\partial r}\frac{\partial}{\partial r},$$

$$\Delta_{C(\Sigma)}u=\frac{1}{r^2}\Delta_\Sigma u(r^{-1}x)+(k-1)r^{-1}\frac{\partial u}{\partial r}+\frac{\partial^2}{\partial r^2}u.$$

Therefore, if we write $x^i=ry^i$, we have

$$\Delta_{C(\Sigma)}x^i=r^{-1}\Delta_\Sigma y^i+(k-1)r^{-1}y^i=r^{-1}\left(\Delta_\Sigma y^i+(k-1)y^i\right).$$

Hence we have the equivalence.

Theorem 2.2. *(Simon) There is no non-flat stable minimal hypercone in* $\mathbb{R}^{n+1}$ *when* $3 \le n+1 \le 7$.

Proof. Let $M = C(\Sigma)$ be a stable minimal hypercone. By Simon's identity

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 - |A|^4,$$

we have

$$|A|L|A| = |A|\Delta_M|A| + |A|^4 = |\nabla A|^2 - |\nabla|A||^2.$$

Let E_i be an orthonormal basis for TM with $E_n = \frac{x}{|x|}$, then

$$a_{ij,n} = \frac{\partial}{\partial r}\left(\frac{1}{r}a_{ij}\left(\frac{x}{|x|}\right)\right) = -\frac{1}{r}a_{ij}(x).$$

$$a_{in} = \langle \nabla_{E_n}E_i, \nu\rangle = -\langle E_i, \nabla_{E_n}\nu\rangle = 0.$$

$$\begin{aligned}
|\nabla A|^2 - |\nabla|A||^2 &= a_{ij,k}^2 - \frac{\sum_{k=1}^n\left(\sum_{i,j=1}^n a_{ij}a_{ij,k}\right)^2}{\sum_{i,j=1}^n a_{ij}^2} \\
&= \frac{\sum_{i,j,r,s,t}(a_{rs}a_{ij,t} - a_{ij}a_{rs,t})^2}{2|A|^2} \\
&\ge \frac{2}{|A|^2}\sum_{i,j,r,s}(a_{jr}a_{ni,s} - a_{ni}a_{jr,s})^2 \\
&= \frac{2}{|A|^2}(a_{jr}a_{ni,s})^2 \\
&= \frac{2}{|A|^2}(a_{jr}a_{is,n})^2 \\
&= \frac{2|A|^2}{|x|^2}.
\end{aligned}$$

Therefore,

$$|A|L|A| = |\nabla A|^2 - |\nabla|A||^2 \ge \frac{2|A|^2}{|x|^2}. \tag{2.3}$$

Plug $|A|f$ into the stability inequality, we have

$$2\int_M \frac{|A|^2}{r^2}f^2 \le \int_M |A|^2|\nabla f|^2.$$

On the other hand, if we set $f = \max\{1, |x|\}^{1-\frac{n}{2}-2\epsilon}|x|^{1+\epsilon}$, then

$$\begin{aligned}
\int_M |A|^2|\nabla f|^2 &= \int_0^\infty r^{n-1}\int_\Sigma \frac{|A|^2}{r^2}|\nabla f|^2 \\
&\le \max\{1+\epsilon, |2 - \frac{n}{2} - \epsilon|\}\int_0^\infty r^{n-1}\int_\Sigma \frac{|A|^2}{r^4}|f|^2 \\
&\le (2-\epsilon)\int_M \frac{|A|^2}{r^2}f^2,
\end{aligned}$$

provided $n \le 7$. Hence we conclude that $|A| = 0$.

Theorem 2.3. (cf.[?Z1]) *If* $u : \mathbb{R}^{n-1} \to \mathbb{R}$ *is an entire solusion of the minimal surface equation for* $n \le 8$*, then* u *is linear.*

Proof. Without loss of generality, we may assume $u(0) = 0$. Denote

$$\Theta(s) = \frac{\text{Area}(B(0,s) \cap \Sigma_u)}{s^{n-1} \cdot \text{Area}(B_1^{n-1})}.$$

By co-area formula, we have

$$\frac{d}{ds}\Theta(s) = \text{Area}(B_1^{n-1})^{-1} s^{-k-1} \int_{\partial B^n(0,s)\cap\Sigma_u} \frac{|x^N|^2}{|x^T|},$$

and

$$\Theta(r) - \Theta(s) = \int_{\Sigma_u \cap (B(0,r)\setminus B(0,s))} \frac{|x^N|^2}{|x|^{n-1}}.$$

$\Theta(s)$ is increasing in s and we denote Θ_∞ to be the limitation as $s \to \infty$. We claim that

$$\Theta_\infty \le \frac{\text{Area}(\partial B_1^n)}{2\text{Area}(B_1^{n-1})}.$$

This claim comes from the following calibration argument. If $u : D \subset \mathbb{R}^{n-1} \to \mathbb{R}$ satisfies the minimal surface equation and $\Sigma \subset D \times \mathbb{R}$ is another surface with $\partial\Sigma = \partial\Sigma_u$, then

$$\text{Area}(\Sigma_u) = \int_{\Sigma_u} \omega = \int_\Sigma \omega \le \text{Area}(\Sigma),$$

where $\omega(X_1, cdots, X_{n-1}) = d\text{Vol}(X_1, \cdots, X_{n-1}, \nu)$ is a closed form and if X_i are unit, then $|\omega(X_1, cdots, X_{n-1})| \le 1$.

Clearly $\Theta_\infty \ge \lim_{s\to 0}\Theta(s) \ge 1$. If the equality holds, then $x^N = 0$, which implies Σ_u is a cone. Since Σ_u is well defined at the origin, it is exactly a hyperplain.

Pick $r_n \to 0$ and consider $\Sigma_\infty = \lim_{n\to\infty} r_n \Sigma_n$. Use stationary varifold theory[?S1], we know that Σ_∞ is a stable minimal cone with constant $\Theta_{\Sigma_\infty}(s) = \Theta_\infty$, which leads to a contradiction to Theorem 2.2.

Theorem 2.4. *(Bombieri, De Giorgi and Giusti) If $n+1 \ge 9$, there exists non-flat complete minimal graphs in $\mathbb{R}^{n+1}$.*

Sketch of Proof.

It suffices to show the case $n = 2m \ge 8$, since $f(x^1, \cdots, x^{n'}) = f(x^1, \cdots, x^n)$ is obviously a solotion to the minmal surface equation when $n' > n$ and f is a solution of $n-$dimensional minimal surface equation. Let $u^1 = ((x^1)^2 + \cdots + (x^m)^2)^{1/2}$ and $u^2 = ((x^{m+1})^2 + \cdots + (x^{2m})^2)^{1/2}$. Assume $f(x)$ is in the form $F(u^1, u^2)$. Rewrite the minimal surface equation, we get

$$0 = (1 + F_2^2)F_{11} - 2F_1F_2F_{12} + (1 + F_1^2)F_{22} + (m-1)\left(\frac{F_1}{u^1} + \frac{F_2}{u^2}\right)(1 + F_1^2 + F_2^2).$$

We denote

$$LF = (1 + F_2^2)F_{11} - 2F_1F_2F_{12} + (1 + F_1^2)F_{22} + (m-1)\left(\frac{F_1}{u^1} + \frac{F_2}{u^2}\right)(1 + F_1^2 + F_2^2).$$

In [?BGG1], Bombieri, De Giorgi and Giusti first construct two function F_1, F_2 with the following properties, in which step $m \ge 4$ is important.

Let

$$T = \{(u^1, u^2) : u^1 \ge 0, u^2 \ge 0\}, T_1 = \{u^2 < u^1\} \cap T, T_2 = \{u^2 > u^1\} \cap T.$$

$LF^1 > 0, LF^2 < 0$ in $\text{Int}(T_1)$, while $LF^1 < 0, LF^2 > 0$ in $\text{Int}(T_2)$. Moreover, $0 < F^1 < F^2$ in T_1, $F^2 < F^1 < 0$ in T_2, $F^1 = F^2$ if $u^1 = u^2$ and $\frac{F^i_j}{u^j}, i, j = 1, 2$ can be extended to T continuously. Denote $f^i(x) = F^i(u^1, u^2)$. One can easily find a sequence g^k from the following Dirichlet problem,

$$\begin{cases} \sum_{i=0}^{2m} \frac{\partial}{\partial x^i}\left(\frac{\frac{\partial g^k}{\partial x^i}}{\sqrt{1+|\nabla g^k|^2}}\right), & in \quad B(0,k), \\ g^k = f^1, & on \quad \partial B(0,k). \end{cases}$$

and with a upper barrier and a lower barrier, we have the estimate

$$f^1 \le g^k \le f^2, x \in B(0,k).$$

To use Arzela-Ascoli theorem, we need the estimate on the gradient of g^k.

Lemma 2.2. *(Bombieri, De Giorgi, Miranda (1969),* cf.[?S2]*) Let U be a C^2 solotion to the minimal surface equation on a ball $B(x_0, \rho) \subset \mathbb{R}^n$, Then*

$$|\nabla u(x_0)| \le C_1 \exp\left(C_2 R^{-1} \sup_{B(x_0,R)}(u - u(x_0))\right),$$

where C_1, C_2 depend only on n.

From the lemma above, we have an estimate for $|\nabla g^k|$ and by Arzela-Ascoli theorem and diagonal method, we can find a subsequence g^{k_j} which uniformly converge to g on any compact set $K \subset \mathbb{R}^n$. Moreover, g satisfies the minimal surface equation and $f^1 \le g \le f^2$. By computing the increasing order of f^i, we know that g cannot be linear.

3. More on Minimal Surface Equation

In the past section, we have used the existence of solution of the Dirichlet problem. In this section, we introduce more details about the existence and regularity of solutions to the minimal surface equation, especially the following Dirichlet problem.

$$\begin{cases} \sum_{i=0}^{2m} \frac{\partial}{\partial x^i}\left(\frac{\frac{\partial u}{\partial x^i}}{\sqrt{1+|\nabla u|^2}}\right), & in \quad U, \\ u = \phi, & on \quad \partial U. \end{cases} \tag{3.1}$$

Lemma 3.1.

$$\text{Area}(\Sigma_u) = \int_U \sqrt{1+|\nabla u|^2} = sup\{\int_U (g_{n+1} + u\text{div}g) : g \in C^1_c(U; \mathbb{R}^{n+1}), ||g||_\infty \le 1\},$$

for all $u \in W^{1,1}(U)$, where $U \subset \mathbb{R}^n$ is a bounded domain.

Let $\mathcal{A}(u) = \text{Area}(\Sigma_u)$ be the area functional, we have several useful properties of $\mathcal{A}$ from the lemma.

Proposition 3.1. *We have the following properties,*

(1) *$u_j \in W^{1,1}(U)$ converge to $u \in W^{1,1}$ in L^1 norm, then*

$$\mathcal{A}(u) \le \liminf_{j\to\infty} \mathcal{A}(u_j).$$

(2) *If $u, v \in W^{1,1}(U)$ with $\nabla u \ne \nabla v$, then for any $t \in (0,1)$, we have*

$$\mathcal{A}(tu + (1-t)v) < t\mathcal{A}(u) + (1-t)\mathcal{A}(v).$$

Theorem 3.1. *Assume U is a bounded domain with ∂U of C^2 continuous class, and $C^{0,1}_k(U,\phi) = \{f \in C^{0,1}(U) : f = \phi, \forall x \in \partial U, [f]_{C^{0,1}} \le k\}$ is non-empty, then there exists $u \in C^{0,1}_k(U,\phi)$ such that*

$$\mathcal{A}(u) = \inf\{\mathcal{A}(u) : u \in C^{0,1}_k(U,\phi)\}.$$

Moreover, if $[u]_{C^{0,1}(U)} < k$, u is exactly a solution to the minimal surface equation.

Proof. It is obviously that $C^{0,1}_k(U,\phi)$ is equicontinuous and uniformly bounded. Applying Arzela-Ascoli theorem and the strictly convex property, we finish the proof.

If $[u]_{C^{0,1}(U)} < k$, for any $v \in C^\infty_c(U)$, there exists $\delta > 0$ such that $u_t = u + tv \in C^{0,1}_k(U)$ for any $t \in (-\delta, \delta)$. By the first variation formula, we know that u is the weak solution to the minimal surface equation.

Definition 3.1. $u \in C^{0,1}_k(U)$ is a *supersolution*(*subsolution*) for $\mathcal{A}$ if for all $v \in C^{0,1}_k(U,)$ with $v \ge u(v \le u)$ on U we have $\mathcal{A}(v) \ge \mathcal{A}(u)$.

Proposition 3.2. *(Weak maximum principle) Let u, v be a supersolution and a subsolution of $\mathcal{A}$ in $C^{0,1}_k(U)$. If $u \ge v$ on the boundary, then $u \ge v$ in U.*

This propostion is directly derived from Definition 3.1 by considering the set $K = \{x \in U : u(x) < v(x)\}$. If we denote

$$\alpha = \sup\{v(y) - u(y) : y \in \partial U\},$$

then we have $v(x) \le u(x) + \alpha$ since $u + \alpha, v$ are supersolution and subsolution respectively, that is,

$$\sup\{v(x) - u(x) : x \in U\} \le \sup\{v(y) - u(y) : y \in \partial U\}.$$

Therefore, we have the following corollary.

Corollary 3.1. *If u_1, u_2 are solutions to (3.1), and it is minimizing, then $u_1 = u_2$.*

To prove the existence of the solution, we use barrier functions.

Definition 3.2. Let $x \in U$, and $d(x) = \mathrm{dist}(x, \partial U)$, $U_\epsilon = \{x \in U : d(x) < \epsilon\}$. Suppose $\phi \in C^{0,1}(\partial U)$. An *upper barrier*(*lower barrier*) v^+ relative to ϕ is a Lipschitz function defined on $\overline{U_\epsilon}$ for some ϵ, with the following properties:

(1) v agrees with ϕ on ∂U,
(2) $v \ge \sup_{\partial U} \phi (v \le \sup_{\partial U} \phi)$ on ∂U_ϵ,
(3) v is a supersolution(subsolution).

Lemma 3.2. *$\phi \in C^{0,1}(\partial U)$, and upper barrier v^+, lower barrier v^- exist. Then $\mathcal{A}$ achieves its minimum on $C^{0,1}(U,\phi)$.*

Proof. Choose k large enough such that $C^{0,1}_k(U,\phi)$ is non-empty. There exists a area minimizing function u by Theorem 3.1. It suffices to show that $[u]_{C^{0,1}(U)} < k$. We clain that $\inf_{\partial U} \phi \le u(x) \le \sup_{\partial U} \phi, \forall x \in U$. This is because $v = \sup_{\partial U} \phi$ is always the area minimizing function on the set $\{x \in U : u(x) \ge \sup_{\partial U} \phi\}$ and by Corollary 3.1, $u = v$. Similar argument holds for $\inf_{\partial U} \phi$. Therefore we have, for $x \in \partial U_\epsilon$,

$$v^- \le u \le v^+,$$

and by weak maximum principle, it holds for $x \in U_\epsilon$. Denote $M = \max([v^+]_{C^{0,1}(U_\epsilon)}, [v^-]_{C^{0,1}(U_\epsilon)})$, and we have for $x \in U_\epsilon$ and $y \in \partial U$,

$$\frac{|u(x) - u(y)|}{|x - y|} \leq M.$$

For $x \in U \setminus U_\epsilon$ and $y \in \partial U$,

$$|u(x) - u(y)| \leq \max\{sup_{\partial U}\phi - u(y), u(y) - \inf_{\partial U}\phi\} \leq \max\{v^+(z) - v^+(y), v^-(y) - v^-(z)\},$$

for any $z \in \partial U_\epsilon$. Hence we have in this case,

$$\frac{|u(x) - u(y)|}{|x - y|} \leq M,$$

and one can choose k large enough such that $[u]_{C^{0,1}(U)} < k$, which completes the proof.

Lemma 3.3. (cf.[?GT]) *∂U is C^2 continuous with non-negative mean curvature, then $d(x)$ is superharmonic in U_ϵ.*

Theorem 3.2. *(H.Jenkins, J.Serrin) Let U be a bounded domain with C^2 continuous boundary, ∂U has non-negative mean curvature and $\phi \in C^2(\partial U)$. Then there exists $u \in C^{0,1}(\overline{U})$ which is a minimizing solution to the equation in the weak sence.*

$$\begin{cases} \sum_{i=0}^{2m} \frac{\partial}{\partial x^i}\left(\frac{\frac{\partial u}{\partial x^i}}{\sqrt{1+|\nabla u|^2}}\right), & in \quad U, \\ u = \phi, & on \quad \partial U. \end{cases}$$

Proof. The proof is from [?BJMR].

We want to construct a barrier function of the form

$$v(x) = \phi(x) + \psi(d(x)),$$

where $\psi \in C^2([0, \epsilon])$ is to be determined. To make v a upper barrier, u should satisfy

$$\mathcal{M}v = \operatorname{div}\left(\frac{\partial_i v}{\sqrt{1+|\nabla v|^2}}\right) \leq 0.$$

Let

$$Lv = (1 + |\nabla v|^2)^{3/2}\mathcal{M}v,$$

we have

$$\begin{aligned} Lv =&(1 + |\nabla v|^2)\Delta\phi - \partial_i\phi\partial_j\phi\partial_{ij}\phi \\ &+ \psi'\left(2\partial_i\phi\partial_j d\Delta\phi + (1 + |\nabla\phi|^2)\Delta d - \partial_i d\partial_j\phi\partial_{ij}\phi - \partial_i\phi\partial_j\phi\partial_{ij}d\right) \\ &+ {\psi'}^2\left(\Delta\phi + 2\partial_i\phi\partial_j d\Delta d - \partial_i d\partial_j d\partial_{ij}\phi\right) + {\psi'}^3\Delta d \\ &+ \psi''\left(1 + |\nabla\phi|^2 - (\partial_i\phi\partial_j d)^2\right). \end{aligned}$$

If we require $\psi' \geq 1, \psi'' < 0$, then

$$Lv \leq \psi'' + C{\psi'}^2.$$

Set $\psi(t) = \frac{\log(1+a\cdot t)}{C}$. If $\psi' \geq 1$ and $\psi(\epsilon) \geq 2\sup_{x\in\overline{U}}|\phi(x)|$, one can choose $\epsilon = a^{-\frac{1}{3}}$ and let ϵ be sufficiently small. Similarly we can construct a lower barrier. The existence comes from Lemma 3.2.

Mean convexity of the boundary is necessary. Gilbarg and Stampacchia showed that, the Dirichlet problem of an elliptic equation in the form

$$\frac{\partial}{\partial x^i}A_i(\frac{\partial u}{\partial x^1},\cdots,\frac{\partial u}{\partial x^n}), A_i\in C^2,$$

can always be solved if U is strictly convex while there exists a elliptic equation in this form cannot be solved if U is not strictly convex. As for minimal surface equation, if ∂U is not mean convex, the Dirichlet problem is not solvable.

Theorem 3.3. (cf.[?BJMR]) *Let U be a bounded domain with C^2 continuous boundary. Suppose there is $x_0\in\partial U$ with $H(x_0)<0$. Then there exists $\phi\in C^2(\partial U)$ such that*

$$\begin{cases}\sum_{i=0}^{2m}\frac{\partial}{\partial x^i}\left(\frac{\frac{\partial u}{\partial x^i}}{\sqrt{1+|\nabla u|^2}}\right), & in \quad U,\\ u=\phi, & on \quad \partial U.\end{cases}$$

admits no solution $u\in C^2(U)$.

Sketch of Proof. Since $H(x_0)<0$, we have $\Delta d>\delta>0$ for $x\in U\cap B(x_0,r)$, where r is sufficiently small. By using the following lemma, one can find a ϕ such that Theorem 3.2 fails.

Lemma 3.4.

$$\mathrm{sup}_{\partial U\cap B(x_0,r)}u<\mathrm{sup}_{\partial U\setminus B(x_0,r)}u+C,$$

for each u satisfies the minimal surface equation, and C depends only on U.

Proposition 3.3. (A Version of Maximum Principle) *Let U be a bounded domain with C^2 continuous boundary, V a non-empty, closed subset of ∂U. $u,v\in C^2(U)\cap C(U\cup V)$ with*

$$\mathcal{M}v\le\mathcal{M}u \text{ in } U \quad \text{and} \quad v\ge u \text{ on } V.$$

Let $U^t=\{x\in U: d(x)\ge t\}$ and ν_t the correspongding unit outward normal vector field along ∂U^t. If for every open set W containing V we have

$$\mathrm{lim}_{t\to0+}\int_{\partial U^t\setminus W}\left(1-\frac{\nabla v\cdot\nu}{\sqrt{1+|\nabla v|^2}}\right)d\mathcal{H}^{n-1}=0,$$

then

$$v\ge u \text{ in } U\cup V.$$

Proof of Lemma 3.4. Let $v(x)=\alpha-\beta d(x)^{\frac12}$ for $x\in U\cap B(x_0,r)$ and $w(x)=\lambda-\mu\mathrm{dist}(x,\partial B(x_0,r))^{\frac12}$ for $x\in U\setminus B(x_0,r)$, and α,β,λ,μ are to be determined.

Set $U_1=U\cap B(x_0,r)$ and $V_1=\overline{U}\cap\partial B(x_0,r)$. We require $\mathcal{M}v\le0$, and $v(x)\ge u(x)$. By computing

$$\left(1-\frac{\nabla v\cdot\nu}{\sqrt{1+|\nabla v|^2}}\right)=1-\frac{\beta}{2\sqrt{d(y)+\beta^2/4}},$$

the condition of Proposition 3.3 holds. By taking $\alpha=\mathrm{sup}_{V_1}u+\beta\sqrt{\mathrm{diam}(U)}$ and $\beta^2>2/\epsilon$, we have

$$u(x)\le v(x)\le\mathrm{sup}_{V_1}u+\beta\sqrt{\mathrm{diam}(U)}.$$

Therefore,

$$\mathrm{sup}_{\partial U\cap B(x_0,r)}u\le\mathrm{sup}_{\overline{U}\cap\partial B(x_0,r)}u+\beta\sqrt{\mathrm{diam}(U)}. \tag{3.2}$$

Set $U_2 = U \setminus B(x_0, r)$, and $V_2 = \partial U \setminus B(x_0, r)$. We require $\mathcal{M}w \le 0$ in U_2 and $w(x) \ge u(x)$ on V_2. Choose $\lambda = \sup_{V_2} u + \mu\sqrt{\mathrm{diam}(U)}$ and $\mu^2 > 2\mathrm{diam}(U)/(n-1)$, similaily we have

$$u(x) \le w(x) \le \sup_{V_2} u + \mu\sqrt{\mathrm{diam}(U)}.$$

Therefore,

$$\sup_{\overline{U}\cap\partial B(x_0,r)} u \le \sup_{\partial U\setminus B(x_0,r)} u + \mu\sqrt{\mathrm{diam}(U)}. \tag{3.3}$$

Hence Lemma 3.4 follows.

Proof of Proposition 3.3. Let $\phi = \max\{0, u - v\}$, then

$$\int_{U^t} (\mathcal{M}v - \mathcal{M}u)\,\phi \le 0.$$

Denote

$$F(X) = \frac{X}{\sqrt{1+|X|^2}},$$

and $V = \{x : u(x) < v(x)\}$ From integration by parts, we have

$$\int_{\partial U^t\setminus V} (F(Dv) - F(Du))\cdot\nu\phi \le \int_{U^t\setminus V} (F(Dv) - F(Du))\cdot D\phi.$$

Let $F = (F^1, \cdots, F^n)$ and

$$a^{ij} = \int_0^1 \frac{\partial F^i}{\partial x^j}(sDu + (1-s)Dv)ds,$$

then we have

$$F^i(Du) - F^i(Dv) = (u_j - v_j)a^i j.$$

Note that $\frac{\partial F^j}{\partial x^i} = \frac{\partial F^i}{\partial x^j}$ and Du, Dv has a uniform bound, so b_{ij} is elliptic with $(b_{ij}) \ge \lambda$ on U^t for each t.

$$\begin{aligned}\int_{U^t} \lambda|D\phi|^2 &\le \int_{U^t} b_{ij}\phi_i\phi_j\\ &= \int_{U^t\setminus V} b_{ij}(u_j - v_j)\phi_i\\ &\le (F(Du) - F(Dv))\cdot\nu\phi\\ &\le \int_{\partial U^t\setminus V} (1 - F(Dv)\cdot\nu)\phi.\end{aligned}$$

By letting $t \to 0$, we obtain $D\phi = 0$, and therefore $v \ge u$.

4. Stability of Complete Minimal Surface in Euclidean Space

In this section, we will first study the structure of complete stable minimal surface in 3-manifolds. We introduce do Carmo's proof [?CP1] of the structure theorem of stability minimal surface in $\mathbb{R}^3$, then in 3-manifolds with non-negative curvature which is proved by D.Fisher-Colbrie and R.Schoen[?CDR1], and in the end O.Chodosh's proof of the theorem when it is in $\mathbb{R}^4$.

Theorem 4.1. *(M.do Carmo and C.K.Peng) Suppose $M \subset \mathbb{R}^3$ is a stable complete immersion minimal surface, then M is flat.*

Sketch of Proof. Denote the immersion $i : \Sigma \to \mathbb{R}^3$. Consider the Riemannian universal covering $\pi : \tilde{\Sigma} \to \Sigma$. Then $i \circ \pi : \tilde{\Sigma} \to \mathbb{R}^3$ is an immersion.

Lemma 4.1. *$\tilde{\Sigma}$ is stable.*

This is because, for each compact domain $\tilde{D}$, there is a non-negative, nontrivial u on $\pi(\tilde{D})$ with $Lu = \Delta_\Sigma u + |A|^2 u = 0$, we can lift u to $\tilde{u}$ on $\tilde{\Sigma}$. Since π is a local isometry, $\tilde{L}\tilde{u} = 0$ with $\tilde{u}$ non-negative.

In this way, we may assume Σ is simply connected, and by uniformization theorem, Σ is conformally equivalent to either $\mathbb{C}$ of the unit disk $\mathbb{D}$ with induced metric $g = \lambda^2(du^2 + dv^2), \lambda \neq 0$.

If Σ is conformally equivalent to $\mathbb{D}$. By the stablility hypothesis, we have

$$\int_\Sigma (u\Delta_\Sigma u + |A|^2u^2)dV \leq 0.$$

In $\mathbb{D}$, $|A|^2 = \frac{2}{\lambda^2}\Delta\log\lambda$, $dV = \lambda^2 dS$, $\Delta_\Sigma = \frac{1}{\lambda^2}\Delta$, where dS denotes the flat area form in $\mathbb{D}$. So the inequality above can be written as

$$\int_{\mathbb{D}} (u\Delta u + u^2\Delta\log\lambda^2)dS \leq 0 \tag{4.1}$$

Set $u = \phi u$, where $\phi = \lambda^{-1}$, we get

$$\begin{aligned} 3\int_{\mathbb{D}} |\nabla\phi|^2u^2dS &\leq \int_{\mathbb{D}} \phi^2|\nabla u|^2dS - 2\int_{\mathbb{D}} \phi u(\nabla u\cdot\nabla\phi)dS \\ &\leq \int_{\mathbb{D}} \phi^2|\nabla u|^2dS + \epsilon\int_{\mathbb{D}} |\nabla\phi|^2u^2dS + \frac{1}{\epsilon}\int_{\mathbb{D}} \phi^2|\nabla u|^2dS, \end{aligned}$$

which implies that $\exists C$, such that for all $u \in C_c^\infty(\mathbb{D})$

$$\int_{\mathbb{D}} |\nabla\phi|^2u^2dS \leq C\int_{\mathbb{D}} \phi^2|\nabla u|^2dS,$$

Since $\nabla_\Sigma = \frac{1}{\lambda}\nabla$, we finally obtain

$$\int_\Sigma |\nabla_\Sigma\phi|^2u^2dV \leq C\int_\Sigma \phi^2|\nabla_\Sigma u|^2dV. \tag{4.2}$$

Fix $a \in (0,1)$, and let u be the cutoff function that $u = 1$ on $B_\Sigma(p, aR)$ and zero ouside $B_\Sigma(p, R)$, $|\nabla u| \leq \frac{C}{(1-a)R}$.

$$\begin{aligned} \int_{B_\Sigma(p,aR)} |\nabla_\Sigma\phi|^2dV &\leq \frac{C}{(1-a)^2R^2}\int_\Sigma \phi^2dV \\ &= \frac{C}{(1-a)^2R^2}\int_{\mathbb{D}} dS \\ &= \frac{\pi C}{(1-a)^2R^2}. \end{aligned}$$

By letting $R \to \infty$, we obtain that $\nabla_\Sigma\phi = 0$, that is, λ is constant, which is a contradiction to the completeness.

If Σ is conformally equivalent to $\mathbb{C}$ with the induced metric $g = \lambda^2(du^2+dv^2), \lambda \neq 0$. Set $\phi = \Delta\log\lambda^2$, and from the stability equation, we have

$$\int_{\mathbb{C}} \phi u^2dS \leq \int_{\mathbb{C}} |\nabla u|^2dS. \tag{4.3}$$

Let $K = -\frac{|A|^2}{2}$ be the Gaussian curvature, on the minimal surface, if $K \neq 0$, we have $\Delta_\Sigma \log(-K) = 4K$, which implies

$$\phi\Delta\phi + \phi^3 - |\nabla\phi|^2 = 0$$

, and then

$$\begin{aligned}\int_{\mathbb{C}} u^2\phi^3 dS &= \int_{\mathbb{C}} u^2|\nabla\phi|^2 dS - \int_{\mathbb{C}} u^2\phi\Delta\phi dS \\ &= \int_{\mathbb{C}} u^2|\nabla\phi|^2 dS + \int_{\mathbb{C}} u^2|\nabla\phi|^2 dS + 2\int_{\mathbb{C}} u\phi\nabla u\cdot\nabla\phi dS\end{aligned}$$

Set $u = u\phi$, from the stability inequality, we get

$$\int_{\mathbb{C}} \phi^3 u^2 dS \leq \int_{\mathbb{C}} \phi^2|\nabla u|^2 dS + \int_{\mathbb{C}} u^2|\nabla\phi|^2 dS + 2\int_{\mathbb{C}} \phi u\nabla u\cdot\nabla\phi dS$$

and therefore,

$$\int_{\mathbb{C}} |\nabla\phi|^2 u^2 dS \leq \int_{\mathbb{C}} \phi^2|\nabla u|^2 dS. \tag{4.4}$$

$$\int_{\mathbb{C}} \phi^3 u^2 dS \leq C\int_{\mathbb{C}} \phi^2|\nabla u|^2 dS. \tag{4.5}$$

Change u into u^3, we get

$$\int_{\mathbb{C}} \phi^3 u^6 dS \leq C\int_{\mathbb{C}} \phi^2 u^4|\nabla u|^2 dS \tag{4.6}$$

$$\leq C_0\left(\int_{\mathbb{C}} \phi^3 u^6 dS\right)^{\frac{2}{3}} \left(|\nabla u|^6\right)^{\frac{1}{3}}. \tag{4.7}$$

and finally,

$$\int_{\mathbb{C}} \phi^3 u^6 dS \leq C_0\int_{\mathbb{C}} |\nabla u|^6 dS. \tag{4.8}$$

Let u be the cutoff function such that $u = 1$ on $B_{\mathbb{C}}(0,R)$, $u = 0$ outside $B_{\mathbb{C}}(0,2R)$ and $|\nabla u| \leq \frac{C}{R}$, then

$$\int_{B_{\mathbb{C}}(0,R)} \phi^3 dS \leq \frac{C}{R^4}.$$

By letting $R \to \infty$, we have $\phi = 0$, which implies Σ is flat.

Several years later, D.Fisher-Colbrie and R.Schoen extended the theorem to the case M is a 3-manifold with non-negative scalar curvature.

Theorem 4.2. *(D.Fisher-Colbrie and R.Schoen) Let M be a complete oriented* 3-*manifold with non-negative scalar curvature and Σ be a complete oriented complete stable minimal surface. Then one of the following holds.*

(1) *Σ is conformally equivalent to S^2 or Σ is a totally geodesic flat torus. Moreover, if $S > 0$, it is a sphere.*
(2) *Σ is conformally equivalent to $\mathbb{C}$ or a cylinder. If Σ is a cylinder with finite absolute total curvature, then it is flat and totally geodesic. If the scalar curvature of M is pointwisely positive, then Σ is not a cylinder with finite total curvature.*

Proof. Consider the orthonormal frame e_1, e_2, e_3 on Σ with e_3 the unit normal vector. $h_{ij} = \langle \nabla_{e_i} e_j, e_3 \rangle$ and we have

$$h_{11} + h_{22} = 0,$$

and for each $f \in C_c^\infty(\Sigma)$

$$\int_\Sigma \left(|\nabla f|^2 - \left(Ric(e_3, e_3) + \sum_{i,j=1}^{2} h_{ij}^2 \right) f^2 \right) \geq 0.$$

From the Gauss-Codazzi equation

$$K_{12} = K + h_{12}^2 - h_{11} h_{22},$$

where K is the intrinsic Gaussian curvature of Σ, we have

$$K = K_{12} - \frac{1}{2} \sum_{i,j=1}^{2} h_{ij}^2.$$

Therefore the stability operator can be written as

$$L = \Delta + \left(S - K + \frac{1}{2} \sum_{i,j=1}^{2} h_{ij}^2 \right).$$

If Σ is compact, we set $f = 1$, and

$$\int_\Sigma K \geq \int_\Sigma S + \frac{1}{2} \sum_{i,j=1}^{2} h_{ij}^2 \geq 0.$$

By Gauss-Bonnet theorem, M has non-negative genus, which implies it is a sphere or a torus.In the latter case, $S = 0$ on Σ and Σ is totally geodesic. Consider the 1^{st} eigenvalue

$$\lambda_1 = \inf\{ \frac{\int_\Sigma \left(|\nabla f|^2 + K f^2 \right)}{\int_\Sigma f^2} \}.$$

By letting f be constant and from the stability, we have $\lambda_1 = 0$. By considering the variation $f + tg$, we have $\Delta f - K f = 0$, and hence K is identically zero.

If Σ is non-compact, we first assume it is simply connected. By Lemma 1.3, there is a positive function g on Σ satisfying

$$\Delta g - K g + \left(S + \frac{1}{2} \sum_{i,j=1}^{2} h_{ij}^2 \right) g = 0.$$

If Σ is not conformally equivalent to the complex plane, it is a disc with a complete metric $g = \lambda^2 (dx^2 + dy^2)$. We lift g to the disc, and the contradiction follows by the next proposition.

Proposition 4.1. *Let $ds^2 = \lambda^2(dx^2 + dy^2)$ be a complete metric on the disc. P is a non-negative function, then there is no positive solution to the elliptic equation*

$$Lu = -\Delta u + K u - P u = 0,$$

where K is the intrinsic Gaussian curvature.

Proof of Proposition 4.1. Consider $v = \lambda^{-1}$, then $\Delta \log v = K$, and we have

$$\Delta v = Kv + \frac{|\nabla v|^2}{v}.$$

Let η be a smooth function supported in $D \subset\subset \Sigma$. We have by calculating

$$\begin{aligned}\lambda_1(D, L) \int_\Sigma (\eta v)^2 &\leq \int_\Sigma (|\nabla(\eta v)|^2 + K(\eta v)^2 - P(\eta v)^2) \\ &\leq \int_\Sigma ((-\eta\Delta\eta)v^2 - \frac{1}{2}\langle \nabla\eta^2, \nabla v^2\rangle - |\nabla h|^2\eta^2) \\ &= \int_\Sigma |\nabla\eta|^2 v^2 - \int_\Sigma |\nabla v|^2\eta^2\end{aligned}$$

Let $\eta \geq 0$ be a cutoff function supported on $B_{ds^2}(0, R)$ which satisfies

$$\eta = 1 \ on \ B_{ds^2}(0, \frac{1}{2}R), |\nabla\eta| \leq \frac{C}{R}.$$

Since $h^2 d\text{Vol} = dx \wedge dy$, we have

$$\lambda_1(B_{ds^2}(0, R), L) \int_\Sigma (\eta v)^2 \leq \frac{C^2\pi}{R^2} - \int_\Sigma |\nabla v|^2\eta^2.$$

By letting R large enough, we have $\lambda_1(B_{ds^2}(0, R), L) < 0$, which implies there is no positive solution.

Thus we know that for general Σ, it is conformally equivalent to $\mathbb{C}$ or a cylinder A. If the latter one holds, the metric can be written as $ds^2 = \lambda^2(dx^2 + dy^2) = \lambda^2|dz|^2$. We set $f = \eta$ be the cutoff function used in the preceding proof. Fix $p \in \Sigma$, then

$$\frac{C^2}{R^2} \int_{B_{|dz|^2}(p,R)} dx \wedge dy \geq \int_\Sigma \left(S - K + \frac{1}{2}\sum h_{ij}^2\right)\eta^2.$$

If $\int_\Sigma |K| < \infty$, by letting $R \to \infty$ and using dorminated convergence theorem, we have

$$\int_\Sigma \left(S + \frac{1}{2}\sum h_{ij}^2\right) \leq \int_\Sigma K.$$

By the Cohn-Vossen inequality(cf.[?CVS]),

$$\int_\Sigma K \leq 0,$$

so Σ is totally geodesic and $S = 0$ on Σ. From the stability, we can find a positive v on Σ with $Lv = 0$. Let $m = \log v$, and by calculating we get

$$C\int_\Sigma |\nabla w|^2\eta^2 \leq \int_\Sigma (\eta^2 K + 4|\nabla\eta|^2).$$

Letting $R \to \infty$, we have

$$C\int_\Sigma |\nabla w|^2 \leq \int_\Sigma K = 0,$$

and hence v is constant and K is identically 0.

If M has non-negative Ricci curvature, with the same arguement, we have Σ is totally geodesic and $Ric(e_3, e_3) = 0$ on Σ. By the Codazzi equation,

$$K = K_{12} = \frac{Ric(e_1, e_1) + Ric(e_2, e_2) - Ric(e_3, e_3)}{2} \geq 0.$$

And from the Cohn-Vossen inequality, K is identically zero.

In R^4, the stability of minimal hypersurface is much harder, because we have no conclusion like the uniformization theorem in higher dimensional manifolds. In 2021, O.Chodosh proved a curvature estimate theorem and solved the Schoen's Conjecture.

Theorem 4.3. *(O.Chodosh) Suppose $\Sigma \subset \mathbb{R}^4$ is a complete, connected, oriented, immerse, stable minimal hypersurface, then it is flat.*

Before we prove the main theorem, we first introduce a lemma about harmonic function on manifold.

Lemma 4.2. *For $n \geq 3$, let Σ^n be a immersed, complete, connected, simply connected, oriented, stable minimal hypersurface in $\mathbb{R}^{n+1}$ with uniformly bounded curvature $|A| \leq C_0$. Then for each $p \in \Sigma$, there exists a harmonic function $u \in C^\infty(\Sigma \setminus \{p\})$ satisfying the following properties:*

1. *u is positive with infimum 0.*
2. *for each compact K such that $p \in \text{Int}(K)$, $\int_{\Sigma \setminus K} |\nabla u|^2 \leq \infty$.*
3. *$u(x) \to 0$ as $d(x, p) \to \infty$.*
4. *$\forall s \geq 0$, $\Omega_s = \{u \geq s\} \cup \{p\}$ is compact.*
5. *if such a s is also a regular value, then $\Sigma_s = u^{-1}(s)$ is a close connected submanifold.*

Remark 4.1. By Sard's theorem, $\mathcal{R} = \{regular\ values\}$ is dense in $(0, \infty)$.

Remark 4.2. Let $\Sigma_s^+ = \{x \in \Sigma_s : |\nabla u|(x) \neq 0\}$, and we define

$$F(s) = \int_{\Sigma_s^+} |\nabla u|^2,$$

and

$$\mathcal{A}(s) = \int_{\Sigma_s^+} |A|^2.$$

One can check that $F(s), \mathcal{A}(s)$ are continuous, and $F(s)$ is Lipschitz on $(0, \infty)$.

Harmonic function on minimal hypersurface can always reflect some essential information. In this section, we will introduce Chodosh's estimate of $F(s)$.

Proposition 4.2. *For any $\phi \in C_c^{0,1}((0, \infty))$, we have the inequality,*

$$\int_0^n \phi(s)^2 \mathcal{A}(s) ds \leq \frac{8\pi}{3} \int_0^\infty \phi(s)^2 ds + \frac{4}{3} \int_0^\infty \phi'(s)^2 F(s) ds. \tag{4.9}$$

Proposition 4.3. *There is a $C > 0$ such that for all $t \in (0, 1)$, the following inequality holds,*

$$F(t) \leq Ct^3 + 4\pi t^2 + \frac{1}{4} t \cdot \text{liminf}_{l \to 0^+} \int_l^t \mathcal{A}(s) ds + \frac{1}{4} t^3 \int_t^1 s^{-2} \mathcal{A}(s) ds. \tag{4.10}$$

Proposition 4.4. *(Schoen, Simon, Yau) For $n \le 7$, if $\Sigma^n \to \mathbb{R}^{n+1}$ is a complete, connected, oriented ,immersed,stable minimal hypersurface, then there is a constant C depending only on n so that*

$$\int_\Sigma |A|^3 u^3 \le C \int_\Sigma |\nabla f|^3, \tag{4.11}$$

for any $f \in C_c^{0,1}(\Sigma)$.

In [?SSY1], the estimate was proved when $p \in [4, 4+\sqrt{8/n})$. Chodosh include the proof when $n = 3$ in his article[?CL1].

Sketch of Proof. Set $e_\epsilon = (|A|^2 + \epsilon)^{\frac{1}{4}}$. Consider $e_\epsilon f$ in the stability inequality we have

$$\int_\Sigma |A|^2 e_\epsilon^2 f^2 \le \int_\Sigma \frac{1}{16} e_\epsilon^{-6} |\nabla |A|^2|^2 f^2 + \frac{1}{2} f e_\epsilon^{-2} \langle \nabla |A|^2, \nabla f\rangle + e_\epsilon^2 |\nabla f|^2. \tag{4.12}$$

Moreover, by Cauchy Schwartz inequality, we have

$$\int_\Sigma |A|^2 e_\epsilon^2 f^2 \le \int_\Sigma \frac{1}{4} e_\epsilon^{-2} |\nabla |A||^2 f^2 + \frac{1}{2} e_\epsilon^{-2} f^2 |\nabla |A||^2 + \frac{1}{2} e_\epsilon^{-2} |A|^2 |\nabla f|^2 + e_\epsilon^2 |\nabla f|^2. \tag{4.13}$$

Multiplying Simon's inequality (cf.[?CM1], Chapter 2.1),

$$\Delta |A|^2 \ge -2|A|^4 + 2\left(1 + \frac{2}{n-1}\right) |\nabla |A||^2, \tag{4.14}$$

by $e_\epsilon^{-2} f^2$ and integrating by part, we obtain

$$\begin{aligned}
&\int_\Sigma 2\left(1+\frac{2}{n}\right) e_\epsilon^{-2} f^2 |\nabla |A||^2 \\
&\le \int_\Sigma 2 e_\epsilon^{-2} f^2 |A|^4 + \frac{1}{2} e_\epsilon^{-6} f^2 |\nabla |A|^2|^2 - e_\epsilon^{-2} f \langle \nabla f, \nabla |A|^2\rangle \\
&\le 2\left(\int_\Sigma e_\epsilon^2 f^2 |A|^2 - \frac{1}{2} e_\epsilon^{-2} f \langle \nabla f, \nabla |A|^2\rangle\right) + \frac{1}{2}\int_\Sigma e_\epsilon^{-6} f^2 |\nabla |A|^2|^2 \\
&\le \frac{5}{8}\int_\Sigma e_\epsilon^{-6} f^2 |\nabla |A|^2|^2 + 2\int_\Sigma e_\epsilon^2 |\nabla f|^2 \\
&\le \frac{5}{2}\int_\Sigma e_\epsilon^{-2} f^2 |\nabla |A||^2 + 2\int_\Sigma e_\epsilon^2 |\nabla f|^2,
\end{aligned}$$

where we use the fact $|A| < e_\epsilon^2$ and $|\nabla |A|^2|^2 = 4|A|^2 |\nabla |A||^2$. Rearrange the equation and we have that

$$\left(\frac{2}{n} - \frac{1}{4}\right) \int_\Sigma e_\epsilon^{-2} f^2 |\nabla |A||^2 \le \int_\Sigma e_\epsilon^2 |\nabla f|^2. \tag{4.15}$$

Combining it with (3.13), we finally arrive at

$$\int_\Sigma |A|^2 e_\epsilon^2 f^2 \le \left(\frac{3}{8} \cdot \frac{4n}{8-n} + \frac{3}{2}\right) \int_\Sigma e_\epsilon^2 |\nabla f|^2. \tag{4.16}$$

Letting $\epsilon \to 0$ (using the dorminated convergence theorem) and replacing f by $f^{3/2}$, we find

$$\int_\Sigma |A|^3 f^3 \le C \int_\Sigma f|A||\nabla f|^2 \le C\left(\int_\Sigma |A|^3 f^3\right)^{1/3} \left(\int_\Sigma |\nabla f|^3\right)^{2/3}. \tag{4.17}$$

therefore the estimate holds when $n \leq 7$.

In Proposition 4.2, we use the logarithmic cutoff function

$$\phi(x) = \begin{cases} 0, & if \quad s \in (0, \epsilon l) \\ 1 - \dfrac{\log(s) - \log(l)}{\log(\epsilon)}, & if \quad s \in [\epsilon l, l) \\ 1, & if \quad s \in [l, t) \\ ts^{-1}, & if \quad s \in [t, 1) \\ t(2-s), & if \quad s \in [1, 2) \\ 0, & if \quad s \in [2, \infty). \end{cases}$$

After caculating, we can obtain that,

$$\int_l^t \mathcal{A}(s)ds + t^2 \int_t^1 s^{-2}\mathcal{A}(s)ds \leq O(t) + \frac{4}{3}\int_t^1 t^2 s^{-4} F(s)ds.$$

Use Proposition 4.3, we get

$$F(t) \leq O(t^2) + \frac{1}{3}t^3 \int_t^1 s^{-4}F(s)ds. \tag{4.18}$$

If there is a sequence $t_j \in (0, 1)$ converging decreasingly to zero so that

$$F(t_j)t_j^2 = sup_{[t_j,1]}F(s)s^{-2},$$

$$\begin{aligned} F(t_j) &\leq O(t_j^2) + \frac{1}{3}t_j^3 \int_{t_j}^1 s^{-2}(s^{-2}F(s))ds \\ &\leq O(t_j^2) + \frac{1}{3}t_j F(t_j) \int_{t_j}^1 s^{-2}ds \\ &= O(t_j^2) + \frac{1}{3}F(t_j)(1 - t_j), \end{aligned}$$

and we finally find that

$$F(t_j) \leq \frac{O(t_j^2)}{\frac{2}{3} + \frac{t_j}{3}},$$

and therefore,

$$F(t) = O(t^2), t \to 0.$$

Proof of Theorem 4.3. Consider $f = \phi \circ u$ and apply Proposition 3.3, we have

$$\begin{aligned}\int_\Sigma |A|^3 \phi(u)^3 &\leq C \int_\Sigma \phi'(u)^3 |\nabla u|^3 \\ &= C \int_0^\infty \phi'(s)^3 \int_{\Sigma_s} |\nabla u|^2 ds \\ &= C \int_0^\infty \phi'(s)^3 F(s) ds \\ &\leq C \int_0^\infty \phi'(s)^3 s^2 ds.\end{aligned}$$

Use the logarithmic cutoff function

$$\phi(x) = \begin{cases} 0, & if \quad s \in (-\infty, R^{-2}) \\ 2 + \dfrac{\log(s)}{\log(R)}, & if \quad s \in [\mathbb{R}^{-2}, R^{-1}) \\ 1, & if \quad s \in [R^{-1}, R) \\ 2 - \dfrac{\log(s)}{\log(R)}, & if \quad s \in [R, R^2) \\ 0, & if \quad s \in [R^2, \infty). \end{cases}$$

We get

$$\begin{aligned}&\int_{R^{-1} \leq u \leq R} |A|^3 \\ &\leq C \int_{R^{-2}}^{R^{-1}} \frac{s^2}{s^3 |\log R|^3} ds + C \int_R^{R^2} \frac{s^2}{s^3 |\log R|^3} ds \\ &= O(|\log R|^{-2}).\end{aligned}$$

By letting $R \to \infty$, we have $|A| = 0$. This implies that such a Σ is flat.

References

[1] Tobias Holck Colding, William P.Minicozzi, "A Course in Minimal Surfaces", *American Mathematical Society*, (2011).

[2] M.do Carmo, C.K.Peng, "Stable Complete Minimal Surfaces in $\mathbb{R}^3$ Are Planes". (1979), 903-906.

[3] R.Schoen, L.Simon, S.T.Yau, Curvature estimates for minimal hypersurfaces". *Acta Math.* (1979), no. 3-4, 275-288.

[4] Otis Chodosh, and Chao Li. "Stable minimal hypersurfaces in $\mathbb{R}^4$". *arXiv preprint* arXiv:2108.11462 (2021).

[5] De Giorgi, Ennio. "Una estensione del teorema di Bernstein". *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze 19.1* (1965): 79-85.

[6] Bombieri, de Giorgi, E.Giusti. "Minimal cones and the Bernstein problem". *Ennio De Giorgi* (1969): 291.

[7] L. Simon. "Lectures on geometric measure theory". *The Australian National University, Mathematical Sciences Institute, Centre for Mathematics and its Applications* (1983)

[8] Xin Zhou. "Lectures on minimal surfaces". Cornell University(2021)

[9] De Giorgi, M.Miranda. "Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche". *Archive for Rat. Mech. and Analysis* (1968)

[10] Begley, T., Jackson, H., Mathews, J., Rockstroh, P. "Existence and Regularity of Solutions to the Minimal Surface Equation". (2013)

[11] Howard Jenkins, James Serrin. The Dirichlet problem for the minimal surface equation in higher dimensions. *Journal Für Die Reine Und Angewandte Mathematik (Crelles Journal)*, (1968): 229.

[12] D.Gilbarg, N.S.Trudinger. ”Elliptic partial differential equations of second order”. *Springer Verlag.* (2001): 224.

[13] L. Simon. ”The minimal surface equation”. *Encyclopaedia of Mathematical Science* (1983): volume 90, 254.

[14] Fischer-Colbrie, Doris, and Richard Schoen. ”The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature.” *Communications on Pure and Applied Mathematics* 33.2 (1980): 199-211.

[15] Cohn-Vossen, Stefan. ”Kürzeste Wege und Totalkrümmung auf Flächen.”. *Compositio Mathematica 2* (1935): 69-133. <http://eudml.org/doc/88615>.